3rd National Congress on Physical Sciences, 29 Sep. – 2 Oct. 2016, Sofia
Section: Mathematical Physics

Multiparameter Quantum Minkowski Space-Time and Quantum Maxwell Equations Hierarchy

V.K. Dobrev

Institute for Nuclear Research and Nuclear Energy,
Bulgarian Academy of Sciences,
72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria

Abstract. Earlier we have proposed new q-Maxwell equations which are the first members of an infinite new hierarchy of q-difference equations. We have used an indexless formulation in which all indices are traded for two conjugate variables, $z, \bar{z}$. We proposed also new q-Minkowski coordinates which together with $z, \bar{z}$ can be interpreted as the six local coordinates of a $SU_q(2,2)$ flag manifold. In the present paper we generalise the main ingredients of this construction to the multiparameter case using the seven-parameter quantum group deformation of $GL(4)$ and $U(gl(4))$ and the four-parameter quantum group deformation of $SL(4)$ and $U(sl(4))$. The main result is the explicit presentation of the multiparameter quantum Minkowski space-time within the corresponding deformed flag manifold.

1 Introduction

Invariant differential equations play a very important role in the description of physical symmetries - recall, e.g., the examples of Dirac, Maxwell equations, (for more examples cf., e.g., [1]). It is important to construct systematically such equations for the setting of quantum groups, where they are expected as (multiparameter) q-difference equations.

In the present paper we consider the construction of deformed multiparameter analogs of some conformally invariant equations, in particular, the Maxwell equations, following the approach of [2]. We start with the classical situation and we first write the Maxwell equations in an indexless formulation, trading the indices for two conjugate variables $z, \bar{z}$. This formulation has two advantages. First, it is very simple, and in fact, just with the introduction of an additional parameter, we can describe a whole infinite hierarchy of equations, which we call the *Maxwell hierarchy*. Second, we can easily identify the variables $z, \bar{z}$ and the four Minkowski coordinates with the six local coordinates of a flag manifold of $SL(4)$ and $SU(2,2)$. Thus, one may look at this as a nice example of unifying internal and external degrees of freedom.

Next we need the deformed analogs of the above constructions. The specifics of the approach of [2] is that one needs also the complexification of the algebra in consideration. Thus for the q - conformal algebra we have used the $U_q(sl(m))$ apparatus of [3] in the case $m = 4$. Thus, in [4] we have proposed new *q-Minkowski coordinates* as part of the appropriate q - flag manifold. Using the corresponding representations and intertwiners of $U_q(sl(4))$ we have also derived an infinite hierarchy of q-Maxwell equations.

In the present paper we generalise the main ingredients of the above construction to the multiparameter case. From [5] we know that the multiparameter quantum group deformation of $GL(m)$, $U(gl(m))$, $SL(m)$, $U(sl(m))$. We apply this for $m = 4$ in order to consider multiparameter deformations of the conformal group, of Minkowski space-time and of Maxwell equations.

2 Classical Setting

It is well known that Maxwell equations

$$\partial^\mu F_{\mu\nu} = J_\nu \,, \qquad \partial^{\mu *} F_{\mu\nu} = 0 \,, \tag{1}$$

or, equivalently

$$\begin{aligned} &\partial_k E_k = J_0 \; (= 4\pi\rho), \\ &\partial_0 E_k - \varepsilon_{k\ell m}\partial_\ell H_m = J_k \; (= -4\pi j_k), \\ &\partial_k H_k = 0 \,, \\ &\partial_0 H_k + \varepsilon_{k\ell m}\partial_\ell E_m = 0 \,, \end{aligned} \tag{2}$$

where $E_k \equiv F_{k0}$, $H_k \equiv (1/2)\varepsilon_{k\ell m}F_{\ell m}$, can be rewritten as

$$\partial_k F^\pm_k = J_0 \;, \quad \partial_0 F^\pm_k \pm i\varepsilon_{k\ell m}\partial_\ell F^\pm_m = J_k \;, \tag{3}$$

$$F^\pm_k \equiv E_k \pm i H_k \;. \tag{4}$$

Not so well known is the fact that the eight equations in (3) can be rewritten as two conjugate scalar equations in the following way:

$$I^+ F^+(z) = J(z,\bar z) \;, \qquad I^- F^-(\bar z) = J(z,\bar z) \;, \tag{5}$$

$$\begin{aligned} I^+ &= \bar z\partial_+ + \partial_v - \frac{1}{2}\Big(\bar z z\partial_+ + z\partial_v + \bar z\partial_{\bar v} + \partial_-\Big)\partial_z \;, \\ I^- &= z\partial_+ + \partial_{\bar v} - \frac{1}{2}\Big(\bar z z\partial_+ + z\partial_v + \bar z\partial_{\bar v} + \partial_-\Big)\partial_{\bar z} \;, \end{aligned} \tag{6}$$

$$\begin{aligned} &x_\pm \equiv x_0 \pm x_3, \quad v \equiv x_1 - i x_2, \quad \bar v \equiv x_1 + i x_2, \\ &\partial_\pm \equiv \partial/\partial x_\pm, \quad \partial_v \equiv \partial/\partial v, \quad \partial_{\bar v} \equiv \partial/\partial \bar v, \end{aligned} \tag{7}$$

$$\begin{aligned} F^+(z) &\equiv z^2(F_1^+ + iF_2^+) - 2zF_3^+ - (F_1^+ - iF_2^+) , \\ F^-(\bar{z}) &\equiv \bar{z}^2(F_1^- - iF_2^-) - 2\bar{z}F_3^- - (F_1^- + iF_2^-) , \\ J(z,\bar{z}) &\equiv \bar{z}z(J_0+J_3) + \bar{z}(J_1 - iJ_2) + z(J_1+iJ_2) + (J_0 - J_3) , \end{aligned} \tag{8}$$

where we continue to suppress the x_μ, resp., $x_\pm, v, \bar{v}$, dependence in F and J. (The conjugation mentioned above is standard and in our terms it is : $I^+ \longleftrightarrow I^-$, $F^+(z) \longleftrightarrow F^-(\bar{z})$.)

It is easy to recover (3) from (5) – just note that both sides of each equation are first order polynomials in each of the two variables z and $\bar{z}$, then comparing the independent terms in (5) one gets at once (3).

Writing the Maxwell equations in the simple form (5) has also important conceptual meaning. The point is that each of the two scalar operators I^+, I^- is indeed a single object, namely it is an intertwiner of the conformal group, while the individual components in (1) - (3) do not have this interpretation. This is also the simplest way to see that the Maxwell equations are conformally invariant, since this is equivalent to the intertwining property.

Let us be more explicit. The physically relevant representations T^χ of the 4-dimensional conformal algebra $su(2,2)$ may be labelled by $\chi = [n_1, n_2; d]$, where n_1, n_2 are non-negative integers fixing finite-dimensional irreducible representations of the Lorentz subalgebra, (the dimension being $(n_1+1)(n_2+1)$), and d is the conformal dimension (or energy). (In the literature these Lorentz representations are labelled also by $(j_1, j_2) = (n_1/2, n_2/2)$.) Then the intertwining properties of the operators in (6) are given by:

$$I^+ \ : \ C^+ \longrightarrow C^0 \ , \quad I^+ \circ T^+ \ = \ T^0 \circ I^+ \ , \tag{9a}$$
$$I^- \ : \ C^- \longrightarrow C^0 \ , \quad I^- \circ T^- \ = \ T^0 \circ I^- \ , \tag{9b}$$

where $T^a = T^{\chi^a}$, $a = 0, +, -$, $C^a = C^{\chi^a}$ are the representation spaces, and the signatures are given explicitly by:

$$\chi^+ = [2,0;2] \ , \quad \chi^- = [0,2;2] \ , \quad \chi^0 = [1,1;3] \ , \tag{10}$$

as anticipated. Indeed, $(n_1, n_2) = (1,1)$ is the four-dimensional Lorentz representation, (carried by J_μ above), and $(n_1, n_2) = (2,0), (0,2)$ are the two conjugate three-dimensional Lorentz representations, (carried by $F_k^\pm$ above), while the conformal dimensions are the canonical dimensions of a current ($d = 3$), and of the Maxwell field ($d = 2$). We see that the variables $z, \bar{z}$ are related to the spin properties and we shall call them 'spin variables'. More explicitly, a Lorentz spin-tensor $G(z,\bar{z})$ with signature (n_1, n_2) is a polynomial in $z, \bar{z}$ of order n_1, n_2, resp. (For more group-theoretical details, cf. [2].)

Formulae (9), (10) are part of an infinite hierarchy of couples of first order intertwiners. Explicitly, instead of (9), (10) we have [2]:

$$I_n^+ \ :\ C_n^+ \longrightarrow C_n^0\ , \quad I_n^+ \circ T_n^+ = T_n^0 \circ I_n^+\ , \tag{11a}$$
$$I_n^- \ :\ C_n^- \longrightarrow C_n^0\ , \quad I_n^- \circ T_n^- = T_n^0 \circ I_n^-\ , \tag{11b}$$

where $T_n^a = T^{\chi_n^a}$, $C_n^a = C^{\chi_n^a}$, and the signatures are:

$$\chi_n^+ = [n+2, n; 2], \quad \chi_n^- = [n, n+2; 2], \quad \chi_n^0 = [n+1, n+1; 3], \quad n \in \mathbb{Z}_+\ , \tag{12}$$

while instead of (5) we have

$$\begin{aligned} I_n^+ F_n^+(z,\bar z) &= J_n(z,\bar z)\ , \\ I_n^- F_n^-(z,\bar z) &= J_n(z,\bar z)\ , \end{aligned} \tag{13}$$

$$\begin{aligned} I_n^+ &= \frac{n+2}{2}\Big(\bar z\partial_+ + \partial_v\Big) - \frac{1}{2}\Big(\bar z z\partial_+ + z\partial_v + \bar z\partial_{\bar v} + \partial_-\Big)\partial_z \\ I_n^- &= \frac{n+2}{2}\Big(z\partial_+ + \partial_{\bar v}\Big) - \frac{1}{2}\Big(\bar z z\partial_+ + z\partial_v + \bar z\partial_{\bar v} + \partial_-\Big)\partial_{\bar z} \end{aligned} \tag{14}$$

while $F_n^+(z,\bar z)$, $F_n^-(z,\bar z)$, $J_n(z,\bar z)$, are polynomials in $z,\bar z$ of degrees $(n+2,n)$, $(n,n+2)$, $(n+1,n+1)$, resp., as explained above. If we want to use the notation with indices as in (1), then $F_n^+(z,\bar z)$ and $F_n^-(z,\bar z)$ correspond to $F_{\mu\nu,\alpha_1,\ldots,\alpha_n}$ which is antisymmetric in the indices μ,ν, symmetric in $\alpha_1,\ldots,\alpha_n$, and traceless in every pair of indices,[1] while $J_n(z,\bar z)$ corresponds to $J_{\mu,\alpha_1,\ldots,\alpha_n}$ which is symmetric and traceless in every pair of indices. Note, however, that the analogs of (1) would be much more complicated if one wants to write explicitly all components. The crucial advantage of (13) is that the operators $I_n^\pm$ are given just by a slight generalization of $I^\pm = I_0^\pm$.

We call the hierarchy of equations (13) the *Maxwell hierarchy*. The Maxwell equations are the zero member of this hierarchy.

Formulae (11)–(13) are part of a much more general classification scheme [2], involving also other intertwining operators, and of arbitrary order. This scheme was adapted to the q-case in [3]. A subset of this scheme are two conjugate infinite two-parameter families of representations which are intertwined by the same operators (14). This is omitted here for the lack of space, cf. [2].

To proceed further we rewrite (14) in the following form:

$$I_n^+ = \frac{1}{2}\Big((n+2)I_1 I_2 - (n+3)I_2 I_1\Big)\ , \tag{15a}$$

$$I_n^- = \frac{1}{2}\Big((n+2)I_3 I_2 - (n+3)I_2 I_3\Big)\ , \tag{15b}$$

[1] In 4D conformal field theory the families of mixed tensors $F_{\mu\nu,\alpha_1,\ldots,\alpha_n}$ appear, e.g., in the operator product expansion of two spin $1/2$ fields [6].

where

$$I_1 \equiv \partial_z\,, \quad I_2 \equiv \bar{z}z\partial_+ + z\partial_v + \bar{z}\partial_{\bar{v}} + \partial_-\,, \quad I_3 \equiv \partial_{\bar{z}}\,. \tag{16}$$

It is important to note that group-theoretically the operators I_a correspond to the right action of the three simple roots of the root system of $sl(4)$, while the operators $I_n^\pm$ are obtained from the lowest possible singular vectors corresponding to the two non-simple non-highest roots [2].

This is the form that we generalize for the deformed case. In fact, we can write at once the general form, which follows from the analysis of [3]:

$$\hat{I}_n^+ = \frac{1}{2}\Big([n+2]_q\hat{I}_1\hat{I}_2 - [n+3]_q\hat{I}_2\hat{I}_1\Big)\,, \tag{17a}$$

$$\hat{I}_n^- = \frac{1}{2}\Big([n+2]_q\hat{I}_3\hat{I}_2 - [n+3]_q\hat{I}_2\hat{I}_3\Big)\,. \tag{17b}$$

Here $\hat{I}_n^\pm$ are obtained from the lowest possible singular vectors of $U_q(sl(4))$, corresponding (as above) to the two non-simple non-highest roots [3].

To proceed further, we should make this form explicit by first generalizing the variables, then the functions and the operators.

3 Multiparameter Quantum Minkowski Space-Time

The variables $x_\pm, v, \bar{v}, z, \bar{z}$ have definite group-theoretical meaning, namely, they are six local coordinates on the flag manifold $\mathcal{Y} = GL(4)/\tilde{B} = SL(4)/B$, where $\tilde{B}, B$ are the Borel subgroups of $GL(4), SL(4)$, respectively, consisting of all upper diagonal matrices. Under a natural conjugation (cf. also below) this is also a flag manifold of the conformal group $SU(2,2)$.

We know from [5] what are the properties of the non-commutative coordinates on the multiparameter $SL_{q,\mathbf{q}}$ flag manifold.

There is a technicality here, namely, that we start from the multiparameter deformation $GL_{q,\mathbf{q}}(m)$ of $GL(m)$ (given by Sudbery [7]) which depends on $(m^2 - m + 2)/2$ parameters q, q_{ij}, $1 \leq i < j \leq m$. (The parametrisation is such that the standard one-parameter deformation is obtained for all $q_{ij} = q$.) Thus, the flag manifold $\tilde{\mathcal{Y}}_{q,\mathbf{q}} = GL_{q,\mathbf{q}}(m)/\tilde{B}_{q,\mathbf{q}}(m)$ depends on the same number of parameters. For $m = 4$ the explicit relations are ($\lambda \equiv q - q^{-1}$):

$$\begin{aligned}
x_+ v &= \frac{q_{23}q_{34}}{q_{24}} v x_+\,, \qquad & \bar{v} x_+ &= \frac{q_{14}}{q_{12}q_{24}} x_+ \bar{v}\,, \\
x_- v &= \frac{q_{13}}{q_{12}q_{23}} v x_-\,, \qquad & \bar{v} x_- &= \frac{q_{13}q_{34}}{q_{14}} x_- \bar{v}\,, \\
\bar{v} v &= \frac{q_{13}q_{34}}{q_{12}q_{24}} v \bar{v}\,, \qquad & \frac{q\,q_{24}}{q_{23}q_{34}} x_+ x_- &= \frac{q_{12}q_{24}}{q\,q_{14}} x_- x_+ + \lambda v \bar{v}\,,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\bar z z = \frac{q_{13}q_{24}}{q_{14}q_{23}} z\bar z \ , \\
&\bar z x_+ = \frac{q_{13}q_{34}}{q_{14}} x_+\bar z \ , \qquad \bar z x_- = \frac{q_{23}q_{34}}{q^2 q_{24}} x_-\bar z + \lambda\bar v \ , \\
&\bar z\bar v = \frac{q_{23}q_{34}}{q_{24}} \bar v\bar z \ , \qquad \bar z v = \frac{q_{13}q_{34}}{q^2 q_{14}} v\bar z \ + \ \lambda x_+ \ , \\
&x_+ z = \frac{q_{14}}{q_{12}q_{24}} z x_+ \ , \qquad x_- z = \frac{q^2 q_{13}}{q_{12}q_{23}} z x_- \ - \ \lambda v \ , \\
&v z = \frac{q_{13}}{q_{12}q_{23}} z v \ , \qquad \bar v z = \frac{q^2 q_{14}}{q_{12}q_{24}} z\bar v - \lambda x_+ \ .
\end{aligned} \tag{19}$$

Thus, in (18) we have a seven-parameter quantum Minkowski space-time.

We note that when all deformation parameter are phases, i.e., $|q| = 1$, $|q_{ij}| = 1$, and in addition holds the following relations:

$$q_{13} = \frac{q_{12}q_{24}}{q_{34}} \ , \qquad q_{14} = \frac{q_{12}q_{24}^2}{q_{23}q_{34}} \ , \tag{20}$$

then the commutation relations (18) and (19) are preserved by an anti-linear anti-involution ω acting as :

$$\omega(x_\pm) = x_\pm \ , \quad \omega(v) = \bar v \ , \quad \omega(z) = \bar z \ . \tag{21}$$

Further, we recall from [5] that the dual quantum algebra $U_{q,\mathbf{q}}(gl(m))$ has the quantum algebra $U_{q,\mathbf{q}}(sl(m))$ as a commutation subalgebra, but not as a co-subalgebra. In order to achieve the complete splitting of $U_{q,\mathbf{q}}(sl(m))$ we have to impose some relations between the parameters, thus the genuine multiparameter deformation $U_{q,\mathbf{q}}(sl(m))$ depends on $(m^2 - 3m + 4)/2$ parameters. Using the same conditions we also ensure that we can restrict from $GL_{q,\mathbf{q}}(m)$ to $SL_{q,\mathbf{q}}(m)$.

Thus, in the case of $m = 4$ for the genuine $U_{q,\mathbf{q}}(sl(4))$ we have four parameters. Explicitly, we achieve this by imposing that the parameters $q_{i,i+1}$ are expressed through the rest as:

$$q_{12} = \frac{q^3}{q_{13}q_{14}} \ , \quad q_{23} = \frac{q^4}{q_{13}q_{14}q_{24}} \ , \quad q_{34} = \frac{q^3}{q_{14}q_{24}} \ . \tag{22}$$

Thus, the four-parameter quantum Minkowski space-time and the embedding quantum flag manifold $\mathcal{Y}_{q,\mathbf{q}}$ are given by (18) and (19) with (22) enforced.

If we would like to enforce also the conjugation (21) then there are more relations between the deformation parameters, namely, we get:

$$q_{12} = q_{23} = q_{34} = \frac{q^2}{q_{14}} \ , \quad q_{13} = q_{24} = q \ . \tag{23}$$

Thus, in this case we have a two-parameter deformation and using the above relations (18) and (19) simplify as follows:

$$\begin{aligned}
&x_+ v = p\, v x_+ \ , \qquad \bar{v} x_+ = p^{-1}\, x_+ \bar{v} \ , \\
&x_- v = p^{-1}\, v x_- \ , \qquad \bar{v} x_- = p\, x_- \bar{v} \ , \\
&\bar{v} v = v \bar{v} \ , \\
&\frac{q}{p}\, x_+ x_- = \frac{p}{q}\, x_- x_+ + \lambda v \bar{v} \ ,
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\bar{z} z = z \bar{z} \ , \\
&\bar{z} x_+ = p\, x_+ \bar{z} \ , \qquad \bar{z} x_- = \frac{p}{q^2}\, x_- \bar{z} \ + \ \lambda \bar{v} \ , \\
&\bar{z} \bar{v} = p\, \bar{v} \bar{z} \ , \qquad \bar{z} v = \frac{p}{q^2}\, v \bar{z} \ + \ \lambda x_+ \ , \\
&x_+ z = p^{-1}\, z x_+ \ , \qquad x_- z = \frac{q^2}{p}\, z x_- - \lambda v \ , \\
&v z = p^{-1}\, z v \ , \qquad \bar{v} z = \frac{q^2}{p}\, z \bar{v} - \lambda x_+ \ ,
\end{aligned} \tag{25}$$

where $p \equiv q^3/q_{14}^2$.

4 Multiparameter Quantum Maxwell Equations Hierarchy

The order of variables hinted in (18),(19) is related to the normal ordered basis of the quantum flag manifold $\mathcal{Y}_{q,\mathbf{q}}$ considered as an associative algebra:

$$\hat{\varphi}_{ijk\ell mn} = z^i\, v^j\, x_-^k\, x_+^\ell\, \bar{v}^m\, \bar{z}^n \ , \quad i,j,k,\ell,m,n \in \mathbb{Z}_+ \ . \tag{26}$$

We introduce now the representation spaces C^χ, $\chi = [n_1, n_2; d]$. The elements of C^χ , which we shall call (abusing the notion) functions, are polynomials in $z, \bar{z}$ of degrees n_1, n_2, resp., and formal power series in the quantum Minkowski variables. Namely, these functions are given by:

$$\hat{\varphi}_{n_1,n_2}(\bar{Y}) = \sum_{\substack{i,j,k,\ell,m,n \in \mathbb{Z}_+ \\ i \leq n_1,\ n \leq n_2}} \mu^{n_1,n_2}_{ijk\ell mn}\, \hat{\varphi}_{ijk\ell mn} \ , \tag{27}$$

where $\bar{Y}$ denotes the set of the six coordinates on $\mathcal{Y}_{q,\mathbf{q}}$. Thus the quantum analogs of $F_n^\pm$, J_n, cf. (13), are:

$$\hat{F}_n^+ = \hat{\varphi}_{n+2,n}(\bar{Y}) \ , \quad \hat{F}_n^- = \hat{\varphi}_{n,n+2}(\bar{Y}) \ , \quad \hat{J}_n = \hat{\varphi}_{n+1,n+1}(\bar{Y}) \ . \tag{28}$$

Using the above machinery we can present a deformed version of the Maxwell hierarchy of equations. First, we mention that the explicit form of the operators I_a in (16) is obtained by the infinitesimal right action of the three simple root generators of $sl(4)$ on the flag manifold $\mathcal{Y}$ (following the procedure of [2]). In the deformed case the right action of $U_{q,\mathbf{q}}(sl(4))$ on $\mathcal{Y}_{q,\mathbf{q}}$ is known from [5], thus, we have:

$$\hat{I}_a = \pi_R(X_a^-) \tag{29}$$

From this we obtain the multi-parameter quantum Maxwell hierarchy of equations by substituting the operators of (29) in (17), i.e., the final result is:

$$\hat{I}_n^+ \, \hat{F}_n^+ = \hat{J}_n \,, \qquad \hat{I}_n^- \, \hat{F}_n^- = \hat{J}_n \,. \tag{30}$$

The reason that we can use (17) is that the multiparameter $U_{q,\mathbf{q}}(sl(4))$ depends only on q as a commutation subalgebra, while the dependence on the other parameters is exhibited only in its co-algebra structure and in the explicit expressions of $\pi_R(X_a^-)$.

Remark: Certainly, as we did in the one-parameter case [4], we would like to present (29) and (30) more explicitly, cf. [8].

Acknowledgments

The author has received partial support from COST Actions MP1210 and MP1405, and from Bulgarian NSF Grant DFNI T02/6.

Bibliography

[1] A.O. Barut and R. Rączka (1980) *Theory of Group Representations and Applications*, II edition, Polish Sci. Publ., Warsaw.

[2] V.K. Dobrev (1988) *Rep. Math. Phys*. **25** 159-181.

[3] V.K. Dobrev (1994) *J. Phys. A* **27** 4841-4857 & 6633-6634, hep-th/9405150.

[4] V.K. Dobrev (1994) *Phys. Lett.* **341B** 133-138 & **346B** (1995) 427.

[5] V.K. Dobrev and P. Parashar (1993) *J. Phys. A* **26** 6991-7002 & Addendum, (1999) **32** 443-444.

[6] V.K. Dobrev, E.H. Hristova, V.B. Petkova and D.B. Stamenov (1974) *Bulg. J. Phys*. **1** 42.

[7] A. Sudbery (1990) *J. Phys. A* **23** L697-L704.

[8] V.K. Dobrev, in preparation.